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Trees with 1-factors and oriented trees

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Abstract

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In this paper we present some results on trees with a 1-factor: generating functions and asymptotics for the number of such trees, labeled, rooted, planted and unlabeled. We show that almost all trees with a 1-factor have nontrivial automorphism groups.

We also exhibit constructive correspondences between trees with a 1-factor and oriented trees, which lead to asymptotics for the number of self-converse oriented trees.

0. Introduction

Given a graph G on n vertices and a sequence $f = (d_1, d_2, \ldots, d_n)$ of nonnegative integers, G has an *f-factor* if there exists a subgraph of G in which the degree of the ith vertex is equal to d_i. In particular, G has a *1-factor* ($f = (1, 1, \ldots, 1)$) if n is even and if there exist n disjoint edges in G. A 1-factor constitutes a complete matching of the vertices of the graph. Classical results regarding f-factors and matchings are due to Tutte [12–14], König [7], Ore [10], Hall [3]. See also [1].

We are concerned here with trees which admit a 1-factor. For short, we will call such trees *matched*. The first observation regarding 1-factors in trees is the following fact.

Fact 0.1. *A finite tree has at most one complete matching.*

This can be easily verified, for example, by induction. The uniqueness of the matching will be useful in many of the proofs following.

In a separate paper, [11], we dealt with enumerative results regarding labeled matched trees: the number of labeled matched trees on $2n$ vertices in $(2n)!\, 2^{n-2}n^{n-2}/n!$; for any fixed integer $d > 1$ and for $n \gg 0$, a vertex selected uniformly at random from the collection of labeled matched trees on $2n$ vertices, will have degree d with probability $\sim(2d-1)/2^d(d-1)!\,e$. The analogous question for all labeled trees was treated by Renyi; there the probability is $\sim 1/((d-1)!\,e)$. In [11] it is also shown that, if $M_{2n,k}$ denotes the number of labeled matched trees on $2n$ vertices having exactly k vertices of degree 1, then the sequence $\{M_{2n,k}\}_k$ is unimodal, in fact, logarithmically concave. Thus, [11] deals with questions on degree distribution in labeled matched trees.

Here we are concerned with planted, rooted, and unlabeled matched trees, identity matched trees, and connections with oriented trees.

In Section 1 we present the generating functions for the types of matched trees mentioned above. Recall that an identity tree is one with trivial automorphism group. In determining these generating functions and, later, asymptotics, one can follow the general approach streamlined in [6]. Therefore we will limit the details of the presentation to those steps where adaptations had to be made to the specifics of matched trees. Recurrence relations are given, allowing the calculation of the number of matched trees of various types: rooted, planted, unlabeled, rooted identity trees, etc.

In Section 2 we establish a connection between matched trees and oriented trees, i.e. trees all of whose edges are assigned, independently, an orientation. We construct bijective correspondences between certain classes of matched trees and oriented trees.

In the last section, using asymptotic formulae for the number of matched trees, we show that, as expected, almost all matched trees have nontrivial automorphisms and, using results from Section 2, we also obtain asymptotics for the number of self-converse oriented trees. An oriented tree is self-converse, if it is isomorphic, as a directed graph, to the oriented tree obtained from it by reversing the orientation of every edge.

In the interest of completeness, we include, with references but no proofs, several characterizations of matched trees.

Theorem 0.2. *Let T be a tree on $2n$ vertices. Then the following are equivalent*:

(i) *T is a matched tree.*

(ii) (Clarke [2, Theorem 3].) *For every vertex v, $T-v$ has precisely one component on an odd number of vertices.*

(iii) (Lovász [9, problem 11.4].) *If B is the adjacency matrix of the bipartite graph T, then* $\det(B)=(-1)^n$.

(iv) (Based on Little's result [8].) *For every subset S of vertices of T, there exists some vertex in T having an odd number of neighbors which lie in S.*

(v) (R. Jamison, private communication.) *If $T_k = \#$ of subtrees of T having k vertices, then $\sum(-1)^k kT_k=0$.*

(vi) *In each maximal clique of the line graph $L(T)$ of T, there is a unique vertex which is not a cutpoint.* (details appear in [11])

1. Generating functions

We will use the following notation: r_{2n}, p_{2n}, and m_{2n} denote, respectively, the number or rooted, planted and unlabeled matched trees on $2n$ vertices.

We first examine the ordinary generating functions (GF) $\mathcal{R}(x)=\sum_{n\geq 0} r_{2n}x^n$, for rooted matched trees, $\mathcal{P}(x)=\sum_{n\geq 0} p_{2n}x^n$, for planted matched trees, and $\mathcal{M}(x)=\sum_{n\geq 0} m_{2n}x^n$, for unlabeled matched trees.

Theorem 1.1. *The following relations hold*:

(a) $\mathcal{R}(x)=x\exp\left\{2\sum_{i\geq 1}\frac{1}{i}\mathcal{R}(x^i)\right\}$;

(b) $\mathcal{P}^2(x)=x\mathcal{R}(x)$;

(c) $\mathcal{M}(x)=\frac{1}{2}\left[\mathcal{R}(x)-\mathcal{R}^2(x)+\mathcal{R}(x^2)+\frac{1}{x}\mathcal{P}(x^2)\right]$.

Proof. (a) Every rooted matched tree T can be decomposed into an ordered pair of forests whose components are rooted matched trees; the first forest has all roots adjacent to the root of T, the second forest has all its roots adjacent to the vertex with which the root of T is paired in the unique matching of T. Now (a) follows routinely.

(b) Consider the same decomposition as in (a) for a planted tree T. Then the first forest discussed above is empty, hence,

$$\mathcal{P}(x)=x\exp\left\{\sum_{i\geq 1}\frac{1}{i}\mathcal{R}(x^i)\right\},$$

which is equivalent to (b).

(c) We sum Otter's formula [5, p. 56], $1=p^*-(q^*-s)$, over all matched unlabeled trees on $2n$ vertices and obtain

$$\mathcal{M}(x)=\mathcal{R}(x)-L(x),$$

where $L(x)$ is the ordinary GF for the number of matched trees rooted at an edge which is not a symmetry edge. The trees in which the root-edge is not part of the matching can be uniquely decomposed into two distinct rooted matched trees; hence, they contribute to $L(x)$

$$\tfrac{1}{2}[\mathcal{R}^2(x)-\mathcal{R}(x^2)]. \tag{1}$$

The trees whose root-edge *is* part of the matching can be decomposed into two distinct planted matched trees overlapping along the edge incident to their roots. Therefore,

$$\frac{1}{2x}[\mathcal{P}^2(x)-\mathcal{P}(x^2)] \tag{2}$$

is the contribution of such trees to $L(x)$. Now summing (1) and (2) gives $L(x)$ and, using (b) we obtain (c). $\square$

Corollary 1.2. *The numbers r_{2n}, p_{2n}, m_{2n} can be calculated from the recurrences:*

$$r_2 = 1, \qquad r_{2n} = \frac{2}{n-1} \sum_{m=1}^{n-1} \left\{ r_{2m} \sum_{d \mid n-m} d r_{2d} \right\}, \quad n \geq 2;$$

$$p_2 = 1, \qquad p_{2n} = \frac{1}{n-1} \sum_{m=1}^{n-1} \left\{ p_{2m} \sum_{d \mid n-m} d r_{2d} \right\}, \quad n \geq 2;$$

and,

$$m_{2n} = \frac{1}{2}\left\{ r_{2n} - \sum_{m=1}^{n-1} \{ r_{2m} r_{2(n-m)} + r_n + p_{n+1} \} \right\}.$$

For odd n, $r_n = p_n = m_n = 0$.

Fig. 1 gives a table of values for $n \leq 12$.

Among matched trees on $2n$ vertices we wish to count now those whose automorphism group is trivial, called identity matched trees. Let $A(x) = \sum a_{2n} x^n$, $R(x) = \sum b_{2n} x^n$ and $P(x) = \sum c_{2n} x^n$ be the ordinary GF's for the number of unlabeled, rooted and planted *identity* matched trees, respectively.

Theorem 1.3.

(a) $$R(x) = x \exp\left\{ 2 \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} R(x^i) \right\},$$

(b) $$P^2(x) = xR(x);$$

(c) $$A(x) = \frac{1}{2}\left[R(x) - R^2(x) - R(x^2) - \frac{1}{x} P(x^2) \right].$$

Proof. (a) As in the proof of Theorem 1.1, an identity rooted matched tree can be decomposed into two distinct forests consisting of identity rooted trees, with the additional property that the trees in each forest are non-isomorphic. Thus,

$$R(x) = x \prod_k (1 + x^k)^{2b_{2k}},$$

which is equivalent to the above claimed relation (a).

(b) In the case of planted matched identity trees, the decomposition from part (a) leads to

$$P(x) = x \prod_k (1 - x^k)^{b_{2k}},$$

which is equivalent to $P^2(x) = xR(x)$.

	matched trees				identity matched trees		
n	rooted $r_{2n}=s_{2n}$	planted $p_{2n}=s_{2n-1}$	unlabeled m_{2n}	oriented σ_{2n}	rooted b_{2n}	planted c_{2n}	unlabeled a_{2n}
1	1	1	1	1	1	1	0
2	2	1	1	1	2	1	0
3	7	3	2	3	5	2	0
4	26	10	5	8	18	7	1
5	107	39	15	27	66	24	4
6	458	160	49	91	266	95	16
7	2 058	702	180	350	1 111	388	64
8	9 498	3 177	701	1 376	4 792	1 650	252
9	44 947	14 830	2 891	5 743	21 124	7 183	1 018
10	216 598	70 678	12 371	24 635	94 888	31 965	4 182
11	1 059 952	342 860	54 564	108 968	432 415	144 502	17 510
12	5 251 806	1 686 486	246 319	492 180	1 994 828	662 241	74 510

Fig. 1.

(c) Since we deal with identity trees, Otter's formula becomes simply $1 = p - q$. Summing this relation over all matched identity trees with $2n$ vertices, we obtain

$$A(x) = U_1(x) - U_2(x),$$

where $U_1(x)$ and $U_2(x)$ are the ordinary GF's for identity matched trees with a root at a vertex and a root at an edge, respectively. Such trees have trivial automorphism groups not only as rooted trees (at a vertex, or at an edge) but also if the root is no longer distinguished. Therefore

$$U_1(x) = R(x) - V_1(x)$$
$$U_2(x) = F(x) - V_2(x),$$

where $V_i(x)$ is the GF for those trees accounted for in $U_i(x)$ which *have* symmetries if the root is no longer distinguished, and $F(x)$ is the GF for matched trees rooted at an edge and having no symmetries *as such*.

Thus,

$$A(x) = R(x) - F(x) - (V_1(x) - V_2(x)), \tag{3}$$

and we now need to determine $F(x)$ and $V_1(x) - V_2(x)$.

In the above we have followed the approach appearing in [5], where the GF for identity trees is treated. In the remainder of the proof the specifics of matched trees will come into play in that trees rooted at an edge will have to be treated separately according to whether or not the root-edge belongs to the (unique) matching of the tree.

Toward $F(x)$, trees rooted at an edge *not* in the matching contribute

$$\tfrac{1}{2}[R^2(x) - R(x^2)], \tag{4}$$

since we form such a tree from two different identity (vertex) rooted matched trees.

The contribution from trees rooted at an edge which is part of the matching is

$$\frac{1}{2x}[P^2(x) - P(x^2)], \tag{5}$$

since each pair of distinct identity planted matched trees yields a different identity edge-rooted matched tree through identification of one's root with the neighbor of the root of the other.

Thus, by adding (4) and (5) we get

$$F(x) = \frac{1}{2}\left[R^2(x) - R(x^2) + \frac{1}{x}P^2(x) - \frac{1}{x}P(x^2)\right]. \tag{6}$$

As it is done in the case of all identity trees, we shall examine the contribution to $V_1(x) - V_2(x)$ of a particular tree T which has symmetries. In our case, T will be a matched tree.

If T has no symmetry line, then its contributions to $V_1(x)$ and $V_2(x)$ cancel.

If T has a symmetry line, say e, then in order for the symmetries of T to be eliminated by vertex or edge rooting, T must consist of two copies of an identity rooted tree t whose roots are joined by e. If t is rooted at any one edge or any one vertex, then the symmetry of T is eliminated; hence T contributes one unit to the appropriate coefficient of $V_1(x) - V_2(x)$. Therefore we must count such matched trees T: if e is not in the matching we have $R(x^2)$ accounting for T; otherwise, we have $(1/x)P(x^2)$. Consequently,

$$V_1(x) - V_2(x) = R(x^2) + \frac{1}{x} P(x^2). \tag{7}$$

Finally, substituting (6) and (7) into (3) and using part (b), we obtain the claimed result. □

From the relations satisfied by the generating functions we deduce the following corollary.

Corollary 1.4. *The numbers b_{2n}, c_{2n} and a_{2n} of identity rooted, planted and unlabeled matched trees can be calculated from the recurrences*:

$$b_{2n} = \frac{2}{n-1} \sum_{m=1}^{n-1} \left\{ b_{2m} \sum_{d \mid n-m} (-1)^{(n-m)/d+1}\, db_{2d} \right\};$$

$$c_{2n} = \frac{1}{n-1} \sum_{m=1}^{n-1} \left\{ c_{2m} \sum_{d \mid n-m} (-1)^{(n-m)/d+1}\, db_{2d} \right\};$$

valid for $n \geq 2$, with initial values $b_2 = c_2 = 1$, and

$$a_{2n} = \frac{1}{2}\left[b_{2n} - \sum_{m=1}^{n-1} b_{2m} b_{2(n-m)} - b_n - c_{n+1} \right].$$

Values for these numbers appear in the table of Fig. 1.

2. A correspondence with oriented trees

Connections between matched trees on $2n$ vertices and oriented trees on n vertices will now be established.

As before, r_{2n}, p_{2n}, and m_{2n} are the number of rooted, planted, and unlabeled matched trees on $2n$ vertices, respectively. Let also d_n, o_n and s_n be respectively the number of rooted, unlabeled, and unlabeled self-converse oriented trees on n vertices.

Theorem 2.1. *For every $n \geq 1$, there exists a bijective correspondence between the rooted matched trees on $2n$ vertices and the oriented rooted trees on n vertices. Consequently, $r_{2n} = d_n$.*

Proof. Let t be a rooted matched tree on $2n$ vertices. We label the vertices of t strictly for convenience in describing the construction of the rooted oriented tree corresponding to it.

Let A be the set consisting of those vertices whose distance to the root is even, and B be the complement of A. Construct a rooted oriented tree T on n vertices as follows: T has one vertex x corresponding to each pair of vertices (a, b) in the unique matching of t; x is the root of t iff x corresponds to the pair of the matching which contains the root of t; if x and y are vertices in T corresponding to the matched pairs (a, b) and (c, d) of t, then x and y are adjacent iff one of $\{a, c\}$, $\{a, d\}$, $\{b, c\}$ or $\{b, d\}$ are adjacent in t; if, say, a and c are adjacent in t, then the edge between x and y is oriented toward y iff a lies in A.

It is easy to check that, this construction is invertible. □

Fig. 2 shows two trees associated under the above bijection.

If the roots are dropped, then the previous construction may fail to be one-to-one because the classes A and B can be interchanged. Thus,

Theorem 2.2. *For every* $n \geqslant 1$,

$$2m_{2n} = o_n + s_n,$$

where m_{2n} = # *unlabeled matched trees on* $2n$ *vertices,* o_n = # *unlabeled oriented trees on* n *vertices, and* s_n = # *unlabeled self-converse oriented trees on* n *vertices.*

Fig. 3 illustrates the construction which proves Theorem 2.2. When the roles of A and B are reversed, converse oriented trees result.

Corollary 2.3. *Let* t *be a matched tree. Then there exist vertices* v *and* w *of* t*, each of degree one, such that the distance between* v *and* w *is odd. In other words, if the vertices of* t *are properly 2-colored, then each color class contains at least one endpoint.*

Proof. Several different proofs are possible, based, e.g., on induction, or on a quantitative version of Hall's condition for the existence of a perfect matching [3]. We give here a proof which seems to us more elegant.

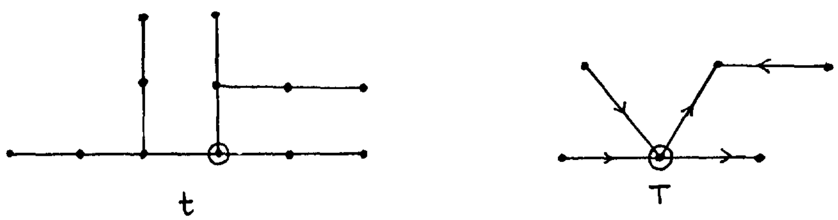

Fig. 2.

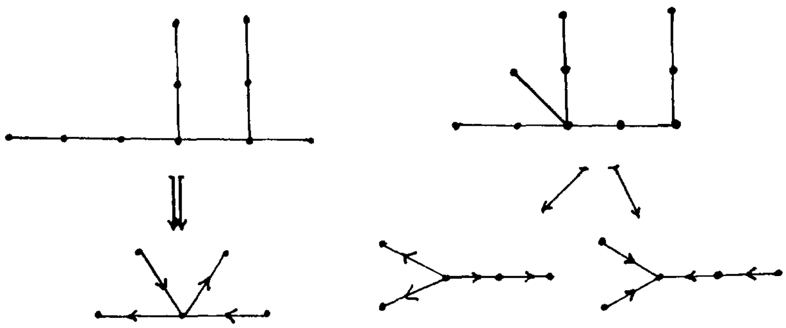

Fig. 3.

With the same notation as used in the proof of Theorem 2.2, let T be an oriented tree corresponding to t. Note that a vertex of T whose indegree or outdegree is zero will correspond to an endpoint in t belonging to B or to A, respectively. Since any oriented tree has vertices of both these kinds (by virtue of being a finite, directed graph with no directed cycles), the corollary follows. □

Remarks. Of course, trees in general do not have the property of Corollary 2.3; for instance, stars $K_{1,m}$ with $m > 1$, have all endpoints at even distance from one another.

There exist matched trees on arbitrarily many vertices having all but one endpoint in the same color class. E.g., see Fig. 4.

Theorem 2.4. *There exist bijective correspondences*:

(a) *between the self-converse oriented trees on* $2n$ *vertices and the rooted matched trees on* $2n$ *vertices*;

(b) *between the self-converse oriented trees on* $2n - 1$ *vertices and the planted matched trees on* $2n$ *vertices.*

Consequently, $s_{2n} = r_{2n}$ *and* $s_{2n-1} = p_{2n}$.

Proof. (a) Let T be a self-converse oriented tree on $2n$ vertices, and let γ be the automorphism between T and its converse T^*. There exists a unique edge $x \to y$ such that $\gamma(x) = y$, $\gamma(y) = x$. Therefore T can be decomposed uniquely into two

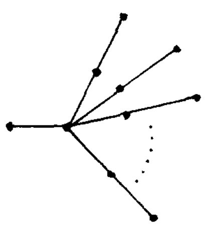

Fig. 4.

rooted oriented trees, t_0 rooted at x, and t_1 rooted at y, such $t_1 = t_0^*$. Let t be the unique rooted matched tree corresponding to t_0 as in Theorem 2.1. Then the claimed bijection associates t with T. Conversely, to recover T from t we construct t_0 as in Theorem 2.1, and direct an edge from the root of t_0 to the root of t_0^*.

(b) If T is a self-converse oriented tree on $2n-1$ vertices, then there is a unique vertex x in T such that $\gamma(x) = x$. Necessarily indegree(x) = outdegree(x). Then T can be decomposed into two oriented trees t_0, t_1 on n vertices, overlapping at x, both rooted at x, with $t_1 = t_0^*$ and x having outdegree zero in t_0. By Theorem 2.1, t_0 will correspond uniquely to a rooted matched tree on $2n$ vertices, and in fact the root will be an endpoint. Thus T corresponds to a unique planted matched tree on $2n$ vertices. This construction is easily seen to be reversible. □

3. Asymptotic results

We first treat the number of rooted matched trees.

Let $\boldsymbol{r}$ be the radius of convergence of the generating function $\mathscr{R}(x)$ appearing in Theorem 1.1(a). From Theorem 2.1, $r_{2n} = d_n \leqslant 2^{n-1}T_n$, where $T_n = \#$ rooted trees on n vertices. Hence, $\mathscr{R}(x) \leqslant \frac{1}{2}\sum_n (2x)^n T_n = \frac{1}{2}T(2x)$, and the GF $T(x)$ is known (see, e.g., [5]) to have radius of convergence $\eta = 0.3383219$. Therefore $\boldsymbol{r} \geqslant (\frac{1}{2})\eta = 0.16916\ldots$.

Using Theorem 1.1(a) and the auxiliary function

$$F(x, y) = x \exp\Big\{2y + 2\sum_{i\geqslant 2}\frac{1}{i}\mathscr{R}(x^i)\Big\} - y,$$

one can follow along the lines of [5, p. 211–212] or [6], and obtain

$$r_{2n} \sim \frac{\boldsymbol{b}_1}{2\sqrt{\pi}} \cdot \boldsymbol{r}^{-n+\frac{1}{2}} n^{-\frac{3}{2}}. \tag{9}$$

The value of $\boldsymbol{b}_1$ is determined by the equation

$$\boldsymbol{b}_1^2 = \frac{1}{2\boldsymbol{r}} + \sum_{i\geqslant 2} \boldsymbol{r}^{i-1}\mathscr{R}'(\boldsymbol{r}^i).$$

The relation obtained in Theorem 1.1(b) shows that $\mathscr{P}(x)$ has the same radius of convergence, $\boldsymbol{r}$, as $\mathscr{R}(x)$ and that $\mathscr{P}(\boldsymbol{r}) = \sqrt{\boldsymbol{r}/2}$. By comparing the expansions of $x\mathscr{R}(x)$ and $\mathscr{P}^2(x)$ in powers of $(\boldsymbol{r}-x)^{\frac{1}{2}}$ one obtains

$$p_{2n} \sim \frac{\boldsymbol{b}_1}{2\sqrt{2\pi}} \boldsymbol{r}^{-n+1} \cdot n^{-\frac{3}{2}}. \tag{10}$$

For unlabeled matched trees the GF $\mathscr{M}(x)$ satisfies (according to Theorem 1.1(c))

$$\mathscr{M}(x) = \frac{1}{2}\Big[\mathscr{R}(x) - \mathscr{R}^2(x) + \mathscr{R}(x^2) + \frac{1}{x}\mathscr{P}(x^2)\Big].$$

Consider and expand in powers of $(\boldsymbol{r}-x)^{\frac{1}{2}}$ the function $G(x)=\mathcal{R}(x)-\mathcal{R}^2(x)=(\frac{1}{4})-(\frac{1}{2}-\mathcal{R}(x))^2$. Then $G(x)$ has radius of convergence equal to $\boldsymbol{r}$ and $G(\boldsymbol{r})=\frac{1}{4}$. It then follows from applying [5, p. 212] that the coefficient of x^n in $G(x)$ is asymptotic to

$$\frac{3\boldsymbol{b}_1\boldsymbol{b}_2}{2\sqrt{\pi}}\boldsymbol{r}^{-n+\frac{3}{2}}n^{-\frac{5}{2}}. \tag{11}$$

Note from (9) and (10) that r_n as well as p_{n+1} are of a lower order of magnitude than $\boldsymbol{r}^{-n}n^{-\frac{7}{2}}$, which is the order of the error term for (11). Therefore,

$$m_{2n}\sim\frac{3\boldsymbol{b}_1\boldsymbol{b}_2}{4\sqrt{\pi}}\cdot\boldsymbol{r}^{-n+\frac{3}{2}}n^{-\frac{5}{2}}. \tag{12}$$

In summary, we have the following theorem.

Theorem 3.1. *The number of matched trees on 2n vertices is as follows:*

(a) *rooted,* $r_{2n}=\dfrac{\boldsymbol{b}_1}{2\sqrt{\pi}}\boldsymbol{r}^{-n+\frac{1}{2}}n^{-\frac{3}{2}}+\mathrm{O}(\boldsymbol{r}^{-n}/n^{\frac{5}{2}})$;

(b) *planted,* $p_{2n}=\dfrac{\boldsymbol{b}_1}{2\sqrt{2\pi}}\boldsymbol{r}^{-n+1}n^{-\frac{3}{2}}+\mathrm{O}(\boldsymbol{r}^{-n}/n^{\frac{5}{2}})$;

(c) *unlabeled,* $m_{2n}=\dfrac{3\boldsymbol{b}_1\boldsymbol{b}_2}{4\sqrt{\pi}}\boldsymbol{r}^{-n+\frac{3}{2}}n^{-\frac{5}{2}}+\mathrm{O}(\boldsymbol{r}^{-n}/n^{\frac{7}{2}})$.

From the above and Theorem 2.4 we have

Corollary 3.2. *The number of self-converse oriented trees is, according to the parity of the number of vertices,*

$$s_{2n}=\frac{\boldsymbol{b}_1}{2\sqrt{\pi}}\boldsymbol{r}^{-n+\frac{1}{2}}n^{-\frac{3}{2}}+\mathrm{O}(\boldsymbol{r}^{-n}/n^{\frac{5}{2}});$$

$$s_{2n-1}=\frac{\boldsymbol{b}_1}{2\sqrt{2\pi}}\boldsymbol{r}^{-n+1}n^{-\frac{3}{2}}+\mathrm{O}(\boldsymbol{r}^{-n}/n^{\frac{5}{2}}).$$

Calculations show that $\boldsymbol{b}_1=0.17709952\cdots$.

Next we shall obtain the following theorem.

Theorem 3.3. *Almost all matched trees have a nontrivial automorphism group.*

Proof. From Theorem 1.3(a) we know that the GF for identity rooted matched trees satisfies

$$R(x)=x\exp\left\{2\sum_{i\geq1}\frac{(-1)^{i+1}}{i}R(x^i)\right\}.$$

Let μ be the radius of convergence for $R(x)$. Certainly, $\mu \geqslant r$. In fact, the same general method which was used in the case of all rooted matched trees yields $R(\mu) = \frac{1}{2}$ and from $R(\mu) < \mu e^{2R(\mu)}$ we get

$$\mu > \frac{1}{2e} = 0.1839 \cdots > r.$$

Thus, $a_{2n} = o(m_{2n})$. $\square$

A more precise result can be obtained regarding the identity matched trees, namely,

$$b_{2n} = \frac{c_1}{2\sqrt{\pi}} \mu^{-n+\frac{1}{2}} n^{-\frac{3}{2}} + \mathrm{O}(\mu^{-n}/n^{\frac{5}{2}})$$

and

$$a_{2n} = \frac{3b_1 b_2}{4\sqrt{\pi}} \mu^{-n+\frac{3}{2}} n^{-\frac{5}{2}} + \mathrm{O}(\mu^{-n}/n^{\frac{7}{2}}),$$

where

$$R(x) = \tfrac{1}{2} - c_1(\mu - x)^{\frac{1}{2}} + c_2(\mu - x) + c_3(\mu - x)^{\frac{3}{2}} + \cdots.$$

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